Free Stein Information

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Our setting: $(\mathcal{A}, \varphi)$ a non-commutative probability space

i.e., $\mathcal{A}$ is a W^*-algebra

φ is a faithful tracial state

E.g. $(L^\infty(\Omega), \mathbb{E})$

$(M_n(\mathbb{C}), \mathrm{tr})$

$(L(\Gamma), \tau)$

$X = (x_1, \dots, x_n)$ an n-tuple of self-adjoint elements of $\mathcal{A}$.

("n" will always be the length of X in this talk)

We will always assume for simplicity that $\mathcal{A} = W^*(X)$.

The law of X is the functional $\mu_X : \mathbb{C}\langle t_1, \dots, t_n \rangle \longrightarrow \mathbb{C}$

$$p \longmapsto \varphi(p(x_1, \dots, x_n))$$

(Note that $\mu_X = \varphi \circ ev_X$.)

We are interested in "regularity properties": when information about μ_X can be passed to information about $W^*(X) = \mathcal{A}$.

Examples:

$\delta(x) = 1 \Rightarrow W^*(x)$ is diffuse [Voiculescu]

$\delta_0(X) > 1 \Rightarrow W^*(X)$ has no Cartan subalgebra, is non-Γ [Voiculescu] and is prime [Ge]

$\Phi^*(X) < \infty \Rightarrow W^*(X)$ is non-Γ [Dabrowski]

$\delta^*(X) = n > 1 \Rightarrow W^*(X)$ is a factor [Dabrowski]

$\delta^*(X) = n \Rightarrow ev_X : \mathbb{C} \langle\!\langle T \rangle\!\rangle \hookrightarrow \mathrm{Aff}(W^*(X))$ [Mai-Speicher-Yin]

"no rational relations"

Entropic properties of the tuple X

Structural properties of $W^*(X)$.

$\mathcal{A}$ – W^*-algebra, φ – faithful tracial state

$X = (x_1, \dots, x_n) \in \mathcal{A}_{sa}^n$ $\quad \mu_X = \varphi \circ ev_X : \mathbb{C}\langle T \rangle \to \mathbb{C}$

The free difference quotients

The free difference quotients are the maps $\partial_j : \mathbb{C}\langle T\rangle \longrightarrow \mathbb{C}\langle T\rangle \otimes \mathbb{C}\langle T\rangle^{op}$ defined by linearity, the Leibniz rule, and the condition $\partial_j(t_i) = \delta_{i=j} 1\otimes 1$.

Examples: $\partial_2(t_1 t_2 t_1 + t_2 t_3 t_2) = t_1 \otimes t_1 + 1 \otimes t_3 t_2 + t_2 t_3 \otimes 1$

$$\partial_j(t_{i_1} \cdots t_{i_n}) = \sum_{k : i_k = j} t_{i_1} \cdots t_{i_{k-1}} \otimes t_{i_{k+1}} \cdots t_{i_n}$$

If $n=1$ and we identify $\mathbb{C}\langle t_1\rangle \otimes \mathbb{C}\langle t_1\rangle^{op} \cong \mathbb{C}[y,z]$,

$$\partial_1 p = \frac{p(y)-p(z)}{y-z} \quad \text{(hence the name)}$$

We also denote $\partial p = (\partial_1 p, \ldots, \partial_n p)$ and let the non-commutative Jacobian be $\mathscr{J}P = (\partial_j p_i)_{i,j=1}^n$.

If $(x_1,\ldots,x_n)$ satisfy no algebraic relation, ∂_j give densely-defined unbounded maps

$$\partial_j : L^2(\mathcal{A}) \supseteq \mathbb{C}\langle X\rangle \longrightarrow \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle^{op} \subseteq L^2(\mathcal{A}\otimes\mathcal{A}^{op})$$
$$p \longmapsto ev_X \circ \partial_j \circ ev_X^{-1}$$

, and similarly we get

$$\partial : L^2(\mathcal{A}) \longrightarrow L^2(\mathcal{A}\otimes\mathcal{A}^{op})^n \quad \text{and} \quad \mathscr{J} : L^2(\mathcal{A})^n \longrightarrow M_n(\mathcal{A}\otimes\mathcal{A}^{op}).$$

We also define the unbounded operator $\mathscr{J}^{\#} : M_n(\mathcal{A}\otimes\mathcal{A}^{op}) \longrightarrow L^2(\mathcal{A})^n$ with domain consisting of those A for which there exists $H \in L^2(\mathcal{A})^n$ so that

$$\langle A, ev_X \circ \mathscr{J}P\rangle_{HS} = \langle H, ev_X P\rangle_2 \quad \forall P \in \mathbb{C}\langle T\rangle^n.$$

If $(x_1,\ldots,x_n)$ satisfy no algebraic relations, $\mathscr{J}^{\#} = \mathscr{J}^{*}$, the adjoint of $\mathscr{J}$. We likewise define $\partial^{\#}$, $\partial_j^{\#}$.

If $A \in \mathrm{dom}\,\mathscr{J}^{\#}$ and $\mathscr{J}^{\#}A = H$, we say A is a free Stein kernel for X relative to H.

$\mathcal{A}$ – W^*-algebra, φ – faithful tracial state

$X = (x_1,\ldots,x_n) \in \mathcal{A}_{sa}^n$ $\quad \mu_X = \varphi \circ ev_X : \mathbb{C}\langle T\rangle \to \mathbb{C}$

Let $\mathbb{1} = \begin{pmatrix} 1\otimes 1 & & 0 \\ & \ddots & \\ 0 & & 1\otimes 1 \end{pmatrix}$. Then Voiculescu's free Fisher information is given by $\Phi^*(X) = \|\mathcal{J}^* \mathbb{1}\|_2^2$ if $\mathbb{1} \in \mathrm{dom}(\mathcal{J}^*)$, and ∞ otherwise. $\Phi^*(X) < \infty$ is a strong regularity condition; we'd like to measure how close it is to being true.

The free Stein irregularity (or "free Stein information") is the quantity

$$\Sigma^*(X) := \mathrm{dist}_{HS}(\mathbb{1}, \mathrm{dom}(\mathcal{J}^*)).$$

Notice immediately that $0 \in \mathrm{dom}(\mathcal{J}^*)$, so $0 \le \Sigma^*(X) \le \sqrt{n} = \mathrm{dist}(\mathbb{1}, 0)$.

The free Stein dimension is the quantity $\sigma(X) := n - \Sigma^*(X)^2 \in [0, n]$.

Example: Single variable x.

Set $\eta_\varepsilon(t) := 2\int \frac{t-s}{(t-s)^2+\varepsilon^2}\, d\mu_x(t)$.

Then $\langle \eta_\varepsilon, p\rangle = 2\iint \frac{t-s}{(t-s)^2+\varepsilon^2} p(t)\, d\mu_x(s)\, d\mu_x(t) = \iint \frac{t-s}{(t-s)^2+\varepsilon^2}(p(t)-p(s))\, d\mu_x(s)\, d\mu_x(t)$

$$= \iint \frac{(t-s)^2}{(t-s)^2+\varepsilon^2} \frac{p(t)-p(s)}{t-s}\, d\mu_x(s)\, d\mu_x(t) = \langle a_\varepsilon, \partial p\rangle$$

where $a_\varepsilon(s,t) = \frac{(t-s)^2}{(t-s)^2+\varepsilon^2}$.

Now $a_\varepsilon \approx \chi_{\{s\neq t\}}$ and so $\|1 - a_\varepsilon\|^2 \longrightarrow \mu_x\otimes\mu_x(\{(t,t)\}) = \sum_t \mu_x(\{t\})^2$.

Hence $\Sigma^*(x)^2 \le \sum_t \mu_x(\{t\})^2$. In fact, equality holds.

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$X = (x_1, \dots, x_n) \in \mathcal{A}_{sa}^n$ $\quad \mu_X = \varphi \circ \mathrm{ev}_X : \mathbb{C}\langle T\rangle \to \mathbb{C}$

$$\partial_j(t_{i_1}\cdots t_{i_n}) = \sum_{k:\, i_k = j} t_{i_1}\cdots t_{i_{k-1}} \otimes t_{i_{k+1}}\cdots t_{i_n}$$

$\partial p = (\partial_1 p, \dots, \partial_n p)$ $\qquad \mathcal{J}P = (\partial_j p_i)_{i,j=1}^n$

$\mathcal{J}^* : M_n(\mathcal{A}\otimes\mathcal{A}^{\circ}) \to L^2(\mathcal{A})^n$ the "adjoint" of $\mathcal{J}$.

$\mathcal{J}^* A = H \Rightarrow A$ is a Stein kernel for X relative to H.

Theorem:

$$\sigma(X) = \dim_{A \otimes A^{op}} \overline{\operatorname{dom} \partial^*} = \frac{1}{n} \dim_{A \otimes A^{op}} \overline{\operatorname{dom} \mathcal{J}^*}$$

Theorem:

$$\Sigma^*(X)^2 + \Sigma^*(Y)^2 \leq \Sigma^*(X, Y)^2$$ with equality if X and Y are free

Idea: If A is a kernel for (X,Y), $A = \left[\begin{array}{c|c} \text{kernel for } X & * \\ \hline * & \text{kernel for } Y \end{array}\right]$ and $\mathbb{1}_{n+m} = \left[\begin{array}{c|c} \mathbb{1}_n & \\ \hline & \mathbb{1}_m \end{array}\right]$,

and $$\left\| \mathbb{1}_n - \boxed{\text{kernel for } X} \right\|_{HS}^2 + \left\| \mathbb{1}_m - \boxed{\text{kernel for } Y} \right\|_{HS}^2 = \left\| \mathbb{1}_{n+m} - \left[\begin{array}{c|c} \text{kernel for } X & 0 \\ \hline 0 & \text{kernel for } Y \end{array}\right] \right\|_{HS}^2 \leq \left\| \mathbb{1}_{n+m} - \left[\begin{array}{c|c} \text{kernel for } X & * \\ \hline * & \text{kernel for } Y \end{array}\right] \right\|_{HS}^2$$

Things of the form $\left[\begin{array}{c|c} \text{kernel for } X & 0 \\ \hline 0 & \text{kernel for } Y \end{array}\right]$ are kernels when X is free from Y.

Theorem: If $S = (s_1, \dots, s_n)$ is a standard free semicircular family free from X,

then $\limsup_{t \searrow 0} t\Phi^*(X + \sqrt{t}S) \leq \Sigma^*(X)^2$

Corollary: $\sigma(X) \leq \delta^*(X)$ $\quad \left(\delta^*(X) := n - \liminf_{t \searrow 0} t\Phi^*(X + \sqrt{t}S) \right)$

(In fact, $\sigma(X) \leq \delta^{\star}(X)$.)

A – W^*-algebra, φ – faithful tracial state

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$$\partial p = (\partial_1 p, \dots, \partial_n p) \qquad \mathcal{J}p = (\partial_j p_i)_{i,j=1}^n$$

$\mathcal{J}^* : M_n(A \otimes A^{op}) \to L^2(A)^n$ the "adjoint" of $\mathcal{J}$.

A is a Stein kernel iff $A \in \operatorname{dom} \mathcal{J}^*$

$$\mathbb{1} = \begin{pmatrix} 1\otimes 1 & & 0 \\ & \ddots & \\ 0 & & 1 \otimes 1 \end{pmatrix}$$

$$\Sigma^*(X) = \operatorname{dist}_{HS}(\mathbb{1}, \operatorname{dom}(\mathcal{J}^*))$$

$$\sigma(X) = n - \Sigma^*(X)^2$$

Theorem: σ is an algebra invariant.

Pf: Since $\sigma(X,0) = \sigma(X)+\sigma(0) = \sigma(X)$, we may assume X and Y have the same number of variables.

Suppose $\mathrm{alg}(X) = \mathrm{alg}(Y)$. Take $F, G \in \mathbb{C}\langle T\rangle$ so that $Y = ev_X F$, $X = ev_Y G$.

Suppose $A \in \mathrm{dom}(\mathcal{J}_X^*)$. Then

$$\langle \mathcal{J}_X^* A, ev_Y P\rangle = \langle \mathcal{J}_X^* A, ev_X P\circ F\rangle = \langle A, ev_X \mathcal{J}(P\circ F)\rangle = \langle A, (ev_Y \mathcal{J}P)\#(ev_X \mathcal{J}F)\rangle = \langle A\#(ev_X\mathcal{J}F)^*, ev_Y \mathcal{J}P\rangle.$$

Hence $A\#(ev_X \mathcal{J}F)^* \in \mathrm{dom}\,\mathcal{J}_Y^*$ and $\mathcal{J}_Y^*(A\#(ev_X\mathcal{J}F)^*) = \mathcal{J}_X^* A$.

Next, since $ev_X T = ev_X G\circ F$ (although $G\circ F$ may not be T),

$$\langle A, \mathbb{1}\rangle = \langle \mathcal{J}_X^* A, ev_X T\rangle = \langle \mathcal{J}_X^* A, ev_X G\circ F\rangle = \langle A, ev_X \mathcal{J}(G\circ F)\rangle$$

Moreover, for $B \in M_n(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle^{op})$ $B^*\#A \in \mathrm{dom}\,\mathcal{J}_X^*$ and

$$\langle A\#(ev_X\mathcal{J}F)^*\#(ev_Y\mathcal{J}G)^*, B\rangle = \langle B^*\#A, (ev_Y\mathcal{J}G)\#(ev_X\mathcal{J}F)\rangle = \langle B^*\#A, ev_X\mathcal{J}(G\circ F)\rangle = \langle B^*\#A, \mathbb{1}\rangle = \langle A, B\rangle$$

Thus $A\#(ev_X\mathcal{J}F)^*\#(ev_Y\mathcal{J}G)^* = A$.

It follows that $\dim_{\mathcal{A}\otimes\mathcal{A}^{op}}(\overline{\mathrm{dom}\,\mathcal{J}_X^*}) \leq \dim_{\mathcal{A}\otimes\mathcal{A}^{op}}(\overline{\mathrm{dom}\,\mathcal{J}_Y^*})$.

By symmetry, $\sigma(X) = \sigma(Y)$.

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$$\Sigma^*(X) = \mathrm{dist}_{HS}(\mathbb{1}, \mathrm{dom}(\mathcal{J}^*))$$

$$\sigma(X) = n - \Sigma^*(X)^2 = \tfrac{1}{n}\dim_{\mathcal{A}\otimes\mathcal{A}^{op}}(\overline{\mathrm{dom}\,\mathcal{J}^*})$$

X, Y free $\Rightarrow \Sigma^*(X,Y)^2 = \Sigma^*(X)^2 + \Sigma^*(Y)^2$ and $\sigma(X,Y) = \sigma(X) + \sigma(Y)$